

Group With Maximum Undirected Edges in Directed Power Graph Among All Finite Non-Cyclic Nilpotent Groups

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Abstract

In [Curtin and Pourgholi, A group sum inequality and its application to power graphs, J. Algebraic Combinatorics, 2014], it is proved that among all directed power graphs of groups of a given order n , the directed power graph of cyclic group of order n has the maximum number of undirected edges. In this paper, we continue their work and we determine a non-cyclic nilpotent group of an odd order n whose directed power graph has the maximum number of undirected edges among all non-cyclic nilpotent groups of order n .

We next determine non-cyclic p -groups whose undirected power graphs have the maximum number of edges among all groups of the same order.

1 Introduction

Many authors studied the directed (undirected) power graphs of finite groups (see [2, 3, 4]). This is an interesting question in this field to determine groups with maximum edges or maximum directed edges in their directed power graphs or undirected power graphs.

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Isaacs et al. in [1] proved that among all groups of order n , the summation of the element orders of the cyclic group of order n is maximum. This is equivalent to this fact that among all groups of order n the number of directed edges in the directed power graph of cyclic group of order n is maximum. Later in [7] the second maximum sum of element orders of finite nilpotent groups was determined.

Curtin et al. in [6] showed that among directed power graphs of finite groups of a given order n , the cyclic group of order n has the maximum number of undirected edges. In another studies [5], they showed that the same is true for undirected power graphs. It is a natural question that which non-cyclic groups have the maximum number of undirected edges in their directed power graphs. In this paper we determine a non-cyclic group G of an odd order such that among all non-cyclic nilpotent groups of order $|G|$ has maximum number of undirected edges in its directed power graph. In the rest of the paper, we determine finite non-cyclic p -groups whose undirected power graphs have the maximum number of edges.

Definition 1. Let G be a finite group. Let $\langle g \rangle$ denote the cyclic subgroup of G generated by $g \in G$.

- (i) The directed power graph $\vec{\mathcal{G}}(G)$ of G is the directed graph whose vertex set is elements of G and for two distinct vertices $x, y \in G$ there is an arc from x to y if and only if $y = x^m$, for some positive integer m ; hence the set of directed edges is $\vec{E}(G) = \{(g, h) \mid g, h \in G, h \in \langle g \rangle - \{g\}\}$ and the set of undirected edges is $\overleftrightarrow{E}(G) = \{\{g, h\} \mid h, g \in G, h \in \langle g \rangle - \{g\} \text{ and } g \in \langle h \rangle - \{h\}\}$.
- (ii) The undirected power graph (or power graph) $\mathcal{G}(G)$ of G is the undirected graph whose vertex set is the elements of G and two vertices being adjacent if one is a power of the other; hence the set of edges is $E(G) = \{(g, h) \mid h, g \in G, g \in \langle h \rangle - \{h\} \text{ or } h \in \langle g \rangle - \{g\}\}$.

Let G be a group. We denote the order of an element a in a group G by $o(a)$. Let ϕ denotes the Euler totient function. Throughout the paper we use C_m to denote the cyclic group of order m . Also, if $n \geq 3$ and p is an odd prime number, then we suppose that $M_{n,p} = \langle a, b \mid b^p = 1 = a^{p^{n-1}}, a^b = a^{1+p^{n-2}} \rangle$.

Let g and h be distinct elements of G . There is an undirected edge between g and h when they generate the same subgroup of G ; hence the number of undirected edges which involve the vertex g is precisely $\phi(o(g)) - 1$. Thus

$$|\overleftrightarrow{E}(G)| = \frac{1}{2} \sum_{g \in G} (\phi(o(g)) - 1),$$

Now let $\phi(G) = \sum_{g \in G} \phi(o(g))$. In order to find non-cyclic nilpotent groups of a given order with the maximum value of $|\overleftrightarrow{E}(G)|$, we should find non-cyclic nilpotent groups with maximum value of ϕ . It was shown in [6] that among all groups of a given order, the cyclic group has the maximum value of ϕ . In this paper, among all non-cyclic nilpotent groups of an odd order, we determine a group with the maximum value of ϕ . We note that if the order of a nilpotent group G is free square, then G is cyclic. Hence, our main result is the following theorem.

Main Theorem. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be a positive odd integer which is not square free and $p_1 < p_2 < \cdots < p_k$ are primes. Set $s = \min\{1, \dots, k\}$ such that $\alpha_s > 1$. Suppose that G is a non-cyclic nilpotent group of order n . Then $\phi(G) \leq \phi(C_{\frac{n}{p_s}} \times C_{p_s})$. Thereore

$$|\overleftrightarrow{E}(C_{\frac{n}{p_s}} \times C_{p_s})| \geq |\overleftrightarrow{E}(G)|,$$

i.e., the directed power graph of $C_{\frac{n}{p_s}} \times C_{p_s}$ has the maximum number of undirected edges among all non-cyclic nilpotent groups of order n .

2 Proof of Main Theorem

For the proof of the main result, we use the following lemma.

Lemma 1. [6, Lemma 3.1] Let U and T be finite groups with $(|U|, |T|) = 1$, and let $G = U \times T$ be the direct product of U and T . Then $\phi(G) = \phi(U)\phi(T)$.

Proposition 1. Among all finite non-cyclic groups of order p^n where p is an odd prime number, the groups $C_{p^{n-1}} \times C_p$ and $M_{n,p}$ have the maximum value of ϕ .

Proof. Let G be a non-cyclic p -group of order p^n . For every non-identity element $g \in G$, we have $\phi(o(g)) = o(g) - \frac{o(g)}{p}$. Therefore

$$\begin{aligned} \phi(G) &= \sum_{g \in G} \phi(o(g)) = \sum_{g \in G - \{e\}} (o(g) - \frac{o(g)}{p}) + 1 \\ &= \sum_{g \in G} o(g) - \sum_{g \in G} \frac{o(g)}{p} + \frac{1}{p} \\ &= (1 - \frac{1}{p}) \sum_{g \in G} o(g) + \frac{1}{p}. \end{aligned}$$

Hence, if $\sum_{g \in G} o(g)$ has the maximum value, $\phi(G)$ has the maximum value, too. It follows from Proposition 2.3 in [7] that $\sum_{g \in G} o(g)$ has the maximum value when $G \cong C_{p^{n-1}} \times C_p$ or $G \cong M_{n,p}$. The proof is complete. \square

Corollary 1. Let G be a non-cyclic group of order p^n where p is an odd prime. Thus

$$|\overleftrightarrow{E}(C_{p^{n-1}} \times C_p)| = |\overleftrightarrow{E}(M_{n,p})| \geq |\overleftrightarrow{E}(G)|.$$

Equality holds when $G \cong C_{p^{n-1}} \times C_p$ or $G \cong M_{n,p}$.

Lemma 2. Let p be a prime number and $m \geq 2$. Then

- (i) $\phi(C_{p^m}) = \phi(C_{p^{m-1}}) + \phi(p^m)^2$;
- (ii) $\phi(C_{p^{m-1}} \times C_p) = p\phi(C_{p^{m-1}}) + (p-1)(p-2)$.

Proof. (i) It is clear that $\phi(C_{p^m}) = \sum_{g \in C_{p^m}} \phi(o(g)) = \sum_{g \in C_{p^{m-1}}} \phi(o(g)) + (p^m - p^{m-1})^2$. The proof of the first part is complete.

(ii) Let $A = \cup_{x \in C_{p^{m-1}}} (x, 0)$, $B = \cup_{0 \neq a \in C_p} (0, a)$ and $C = \cup_{0 \neq a \in C_p} \cup_{0 \neq x \in C_{p^{m-1}}} (x, a)$. Obviously they are a partition for $C_{p^{m-1}} \times C_p$ and so $C_{p^{m-1}} \times C_p = A \cup B \cup C$. It is clear that $\phi(A) = \phi(C_{p^{m-1}})$ and $\phi(B) = \phi(C_p) - 1$. If $(x, a) \in C$, then $o((x, a)) = o(x)$. Since for each $a \in C_p$ we have $\phi(\cup_{0 \neq x \in C_{p^{m-1}}} (x, a)) = \phi(C_{p^{m-1}}) - 1$, this yields that $\phi(C) = \sum_{0 \neq a \in C_p} \sum_{0 \neq x \in C_{p^{m-1}}} \phi(o(x)) = (p-1)(\phi(C_{p^{m-1}}) - 1)$. Thus we have $\phi(C_{p^{m-1}} \times C_p) = \phi(A) + \phi(B) + \phi(C) = \phi(C_{p^{m-1}}) + \phi(C_p) - 1 + (p-1)(\phi(C_{p^{m-1}}) - 1)$. We also have $\phi(C_p) = \sum_{x \in C_p} \phi(o(x)) = (p-1)^2 + 1$; hence, $\phi(C_{p^{m-1}} \times C_p) = p\phi(C_{p^{m-1}}) + (p-1)(p-2)$. \square

Lemma 3. *Let p be a prime number and $m \geq 2$. Then*

$$(p-2)\phi(C_{p^{m-1}} \times C_p) < \phi(C_{p^m}) < p\phi(C_{p^{m-1}} \times C_p).$$

Proof. It follows from Lemma 2 that

$$\phi(C_{p^m}) = \phi(C_{p^{m-1}}) + p^{2m-2}(p-1)^2.$$

According to Lemma 2.5 in [5], we have $\phi(C_{p^{m-1}}) = \frac{p^{2m-2}(p-1) + 2}{p+1}$. Therefore

$$\begin{aligned} \phi(C_{p^m}) &= \phi(C_{p^{m-1}}) + p^{2m-2}(p-1)^2 \leq \phi(C_{p^{m-1}}) + \frac{(p^{2m-2}(p-1) + 2)(p^2 - 1)}{p+1} \\ &= \phi(C_{p^{m-1}}) + \phi(C_{p^{m-1}})(p^2 - 1) = p^2\phi(C_{p^{m-1}}). \end{aligned}$$

By Lemma 2 we have

$$p^2\phi(C_{p^{m-1}}) \leq p\phi(C_{p^{m-1}} \times C_p),$$

and it completes the proof of the right side of the inequality.

By Lemma 2 we have

$$\begin{aligned} \frac{\phi(C_{p^m})}{\phi(C_{p^{m-1}} \times C_p)} &= \frac{\phi(C_{p^{m-1}}) + \phi(p^m)^2}{p\phi(C_{p^{m-1}}) + (p-1)(p-2)} > \frac{\phi(p^m)^2}{p(\phi(C_{p^{m-1}}) + p)} \\ &= \frac{p^{2m-3}(p-1)^2}{\phi(C_{p^{m-1}}) + p} > \frac{p^{2m-3}(p-1)^2}{p^{m-1}(p^{m-1} - p^{m-2}) + p^{2m-3}} \\ &= \frac{(p-1)^2}{p} > p-2, \end{aligned}$$

and we get the result. \square

Corollary 2. *Let p and q be two prime numbers where $p < q$ and $t, m \geq 2$. Then*

$$\frac{\phi(C_{p^m})}{\phi(C_{p^{m-1}} \times C_p)} < \frac{\phi(C_{q^t})}{\phi(C_{q^{t-1}} \times C_q)}.$$

Proof. The right side of the inequality is greater than $q - 2$ by previous lemma and since $p < q$ are odd primes, we have $p \leq q - 2$. According to previous lemma, the left side is less than p . This completes the proof. \square

Proof of the Main Theorem. Let G be a non-cyclic nilpotent group of order n . Since G is a nilpotent group of order n , we have $G \cong P_1 \times P_2 \times \cdots \times P_k$, where P_i is the unique Sylow p_i -subgroup of G of order $p_i^{\alpha_i}$. Therefore $\phi(G) = \phi(P_1)\phi(P_2) \cdots \phi(P_k)$. Because we assume that G is not cyclic, at least one of the Sylow subgroups, say P_j , is not cyclic. Hence we have $\phi(P_j) \leq \phi(C_{p_j^{\alpha_j-1}} \times C_{p_j})$ by Proposition 1. It follows from Lemma 3.6 in [6] that $\phi(P_i) \leq \phi(C_{p_i^{\alpha_i}})$, for $1 \leq i \leq k$. Therefore, we have

$$\begin{aligned} \phi(G) &\leq \phi(C_{p_1^{\alpha_1}}) \cdots \phi(C_{p_{j-1}^{\alpha_{j-1}}}) \phi(C_{p_j^{\alpha_j}}) \phi(C_{p_{j+1}^{\alpha_{j+1}}}) \cdots \phi(C_{p_k^{\alpha_k}}) \\ &= \phi(C_{\frac{n}{p_j^{\alpha_j}}}) \phi(C_{p_j^{\alpha_j-1}} \times C_{p_j}) \\ &= \frac{\phi(C_n)}{\phi(C_{p_j^{\alpha_j}})} \phi(C_{p_j^{\alpha_j-1}} \times C_{p_j}) \\ &\leq \frac{\phi(C_n) \phi(C_{p_s^{\alpha_s-1}} \times C_{p_s})}{\phi(C_{p_s^{\alpha_s}})} = \phi(C_{\frac{n}{p_s}} \times C_{p_s}), \end{aligned}$$

where the last inequality holds by Corollary 2. So the proof is complete.

Corollary 3. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be a positive odd integer which is not square free and $p_1 < p_2 < \cdots < p_k$ are primes. Set $s = \min\{1, \dots, k\}$ such that $\alpha_s > 1$. Then among directed power graph of non-cyclic nilpotent groups of order n , the directed power graph $\vec{G}(C_{\frac{n}{p_s}} \times C_{p_s})$ has the maximum number of undirected edges.

Proposition 2. Let G be a non-cyclic p -group of order p^n whose undirected power graph has the maximum number of edges among all non-cyclic p -groups of order p^n . Then

- (i) if p is odd, then $G \cong C_{p^{m-1}} \times C_p$ or $G \cong M_{n,p}$;
- (ii) if $p = 2$ and $n \neq 3$, then $G \cong C_{2^{n-1}} \times C_2$;
- (iii) if $p^n = 8$, then $G \cong Q_8$.

Proof. It follows from Theorem 4.2 in [4] that

$$|E(G)| = \frac{1}{2} \sum_{g \in G} (2o(g) - \phi(o(g)) - 1).$$

Since G is a finite p -group, for every non-identity element $g \in G$ we have $\phi(o(g)) = o(g) - \frac{o(g)}{p}$. Therefore

$$\begin{aligned} |E(G)| &= \frac{1}{2} [\sum_{g \in G - \{e\}} (2o(g) - (o(g) - \frac{o(g)}{p})) - |G| + 1] \\ &= \frac{1}{2} [\sum_{g \in G} o(g) + \frac{1}{p} \sum_{g \in G} o(g) - |G| - \frac{1}{p}]. \end{aligned}$$

If p is an odd prime number, then $E(G)$ has the maximum value when $\Sigma_{g \in G} o(g)$ has its maximum value, too. By Proposition 3.2 in [7], we know that if a non-cyclic p -group G of order p^n has the maximum value of $\Sigma_{g \in G} o(g)$, then $G \cong C_{p^{n-1}} \times C_p$ or $G \cong M_{n,p}$. It completes the proof of (i).

If $p = 2$ and $n \neq 3$, then according to Proposition 3.2 in [7], $\Sigma_{g \in G} o(g)$ has the maximum value when $G \cong C_{2^{n-1}} \times C_2$.

If $p^n = 8$, then by Proposition 3.2 in [7], $\Sigma_{g \in G} o(g)$ has the maximum value when $G \cong Q_8$. \square

Question 1. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be a positive odd integer which is not square free and $p_1 < p_2 < \cdots < p_k$ are primes. Set $s = \min\{1, \dots, k\}$ such that $\alpha_s > 1$. Among non-cyclic nilpotent groups of order n , does the undirected power graph $\mathcal{G}(C_{\frac{n}{p_s}} \times C_{p_s})$ have the maximum number of edges?

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